UNIVERSITÉ DU LUXEMBOURG
ANALYSE 1
2015-2016

EXERCISE SHEET 3

2.1. Compute the explicit formula for the general term of the following sequences:

2.1.1. $x_{n+1} = 5x_n$

2.1.2. $x_{n+1} = 2x_n + 3$

2.1.3. $x_{n+1} = 2x_n - n^2$

2.1.4. $x_{n+1} = 5x_n - 2^n$

2.2. Compute the limit of the following sequences

2.2.1. $x_{n+1} = -3x_n + 2;$

2.2.2. $x_{n+1} = \frac{7}{3}x_n - \frac{2}{3}x_{n-1};$

2.2.3. $\begin{cases} a_{n+1} = 4a_n - b_n \\ b_{n+1} = 2a_n + b_n \end{cases}$

2.3. Given the following sequence defined by recursion

$$x_{n+1} = 3x_n + 7x_{n-1} \qquad (*)$$

decide if the following statement is true or false: for every $i \neq j \in \mathbb{N}$ and for every $a \neq b \in \mathbb{R}$, there exists a sequence x_n satisfying $(*)$ such that $x_i = a$ and $x_j = b$.

2.4. Let a_n be a sequence of integer numbers such that $a_{2015} \neq 0$ and which satisfies the following relation:

$$2a_{n+2} - 7a_{n+1} + 3a_n = 0$$

Determine the value $\frac{a_{25}}{a_{22}}$.

2.5. Let a_n be a sequence of real numbers such that

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_{n+2} = 2(a_{n+1} - a_n) \end{cases}$$

Determine the biggest power of 2 which divides a_{2015}.

2.6. Let $a, b \in \mathbb{R}_+$. Define the following sequences

$$\begin{cases} u_0 = a \\ v_0 = b \\ u_{n+1} = \frac{v_n + u_n}{2} \\ v_{n+1} = \sqrt{u_n v_n} \end{cases}$$

Prove that the sequences u_n and v_n converge to the same limit.

2.7. Let $a, b \in \mathbb{R}_+$. Define the following sequences

$$\begin{cases} u_0 = a \\ v_0 = b \\ u_{n+1} = \frac{v_n + u_n}{2} \\ \frac{1}{v_{n+1}} = \frac{1}{2}\left(\frac{1}{u_n} + \frac{1}{v_n}\right) \end{cases}$$

Prove that the sequences u_n and v_n converge to the same limit and compute it.

2.8. Compute the limit of the following sequences

2.8.1.

$$\begin{cases} x_{n+1} = \frac{2x_n + 5}{n} \\ x_1 = 2015 \end{cases}$$

2.8.2.

$$\begin{cases} x_{n+1} = \frac{x_n}{\sqrt{n+3}} \\ x_0 = 2015 \end{cases}$$

2.9. Compute the limit of the following sequences

2.9.1. $x_n = \sum_{k=n}^{2n} \frac{1}{k^2}$

2.9.2. $x_n = \sum_{k=n}^{2n} \frac{1}{\sqrt{k}}$

2.9.3. $x_n = \sum_{k=n^2}^{2n^2} \frac{1}{\sqrt[5]{k}}$

2.10. Prove that there exists the limit of the following sequence (you're NOT asked to compute it)

$$x_n = \sum_{k=n}^{2n} \frac{1}{k}$$

2.11. **. Let $A \subset \mathbb{R}$ a non-empty, upper-bounded set. Show that there exists an increasing sequence a_n of elements of A, such that

$$\lim_{n \to \infty} a_n = \sup(A)$$